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Dynamic Thermoelastic Coupling Effects in a Rod

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Nomenclature

A = cross-sectional area of the rod
C = specific heat
c = coefficient of externally induced viscous damping (friction with air)
E = Young's modulus
F = given distributed load (force per unit length)
L = length of rod
Q = given incident heat flux (power per unit volume)
T_0 = reference temperature in which the rod is undeformed
$\mathcal{T}$ = temperature above reference temperature T_0
$\mathcal{U}$ = axial displacement
α = coefficient of thermal expansion
δ = thermoelastic coupling coefficient
κ = thermal conductivity
ρ = mass density
ω = temporal frequency

Introduction

THE prediction of the dynamic response of thermally loaded elastic structures is of considerable importance in aerospace engineering and has been the subject of intensive research. In most cases the uncoupled equations of thermoelasticity have been considered, where the temperature field induces strains in the elastic body, but the elastic deformation is assumed not to affect the temperature field in turn. However, in some works the thermoelastic coupling effects have been included. The underlying principles and governing equations of linear coupled thermoelasticity are reviewed, e.g., by Nowacki.[1]

The thermoelastic coupling term in the heat equation acts like a thermal source that is proportional to the strain rate. Thus, a nonzero strain rate at a point in the material produces some change in the heat flux. This coupling effect is a linear dynamic effect. There are also additional types of thermoelastic coupling effects that are inherently nonlinear. One such effect occurs when the incident heat flux depends on the orientation of the structural members, as typical to space structure applications; see Givoli and Rand[2] and Rand and Givoli.[3] However, in this Note we shall be concerned with linear thermoelastic coupling.

Analytic solutions to coupled thermoelastic problems were considered by various authors; e.g., see Boley and Tolins,[4] Bahar and Hetnarski,[5] Takeuti et al.,[6] and Atkinson.[7] Numerical methods for coupled thermoelastic problems were also proposed. Some of the recent works are by Kasti et al.[8] and Farhat et al.[9]

Although it is widely known that thermoelastic coupling effects may be neglected in many applications, they may become important in some special cases. The purpose of this Note is to characterize the conditions under which thermoelastic coupling effects in a rod become relatively significant and to discuss the importance of these effects in aerospace applications.

Time-Harmonic Coupled Thermoelasticity

We consider the coupled axial heat flow and elastic deformation in a rod of length L. The thermal and elastic material behavior is assumed to be linear and the deformation to be small. We denote the axial coordinate by x and time by t.

To write the governing equations in a nondimensional form, we define the nondimensional variables and parameters,

$$\xi = x/L, \qquad \tau = \kappa t/(\rho C L^2)$$
$$\bar{\mathcal{T}}(\xi,\tau) = \mathcal{T}/T_0, \qquad \bar{\mathcal{U}}(\xi,\tau) = \mathcal{U}/L \tag{1}$$

$$\bar{\delta} = \frac{\delta}{T_0 \rho C}, \qquad \bar{Q} = \frac{L^2 Q}{\kappa T_0}, \qquad \bar{\rho} = \frac{\kappa^2 \rho}{(\rho C)^2 E L^2}$$
$$\bar{c} = \frac{\kappa c}{\rho C E}, \qquad \bar{\alpha} = T_0 \alpha, \qquad \bar{F} = \frac{LF}{EA} \tag{2}$$

Then the thermoelastic equations in nondimensional form are

$$\bar{\mathcal{T}}_\tau = \bar{\mathcal{T}}_{\xi\xi} - \bar{\delta}\bar{\mathcal{U}}_{\xi\tau} + \bar{Q}(\xi,\tau) \tag{3}$$

$$\bar{\rho}\bar{\mathcal{U}}_{\tau\tau} + \bar{c}\bar{\mathcal{U}}_\tau = \bar{\mathcal{U}}_{\xi\xi} - \bar{\alpha}\bar{\mathcal{T}}_\xi + \bar{F}(\xi,\tau) \tag{4}$$

These equations hold in the unit interval $0 < \xi < 1$. The subscripts τ and ξ indicate partial differentiation. In what follows we shall concentrate on the thermally driven case where $\bar{F} \equiv 0$ in Eq. (4), as typical for example in space structure applications.

We now apply the Fourier transform in τ to Eqs. (3) and (4). This yields the thermoelastic equations in terms of ξ alone for a fixed temporal frequency ω. (We remark that it is also possible to apply the Laplace transform in τ, but this turns out to lead to more complicated mathematical expressions.) We define the nondimensional frequency $\bar{\omega}$ by requiring that $\omega t = \bar{\omega}\tau$, which leads to $\bar{\omega} = \rho C L^2 \omega/\kappa$. Denoting by $q(\xi)$, $T(\xi)$, and $u(\xi)$ the complex-valued Fourier transforms corresponding to $\bar{Q}$, $\bar{\mathcal{T}}$, and $\bar{\mathcal{U}}$, respectively, we obtain the following two equations for T and u:

$$T'' - i\bar{\omega}T - i\bar{\omega}\bar{\delta}u' + q = 0 \tag{5}$$

$$u'' + (\bar{\omega}^2\bar{\rho} - i\bar{\omega}\bar{c})u - \bar{\alpha}T' = 0 \tag{6}$$

where a prime indicates differentiation with respect to ξ. Equations (5) and (6) are accompanied by appropriate boundary conditions to complete the statement of the problem. Generally, unless $q(\xi)$ in Eq. (5) is especially simple, a particular analytic solution may be

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hard to find, and Eqs. (5) and (6) must be solved numerically. The solution to the original time-dependent problem may be obtained by applying the inverse Fourier transform to the solution of Eqs. (5) and (6).

Solution by the Galerkin Method

The thermoelastic problem will now be solved using the Galerkin method. Let $[\phi_n(\xi)]$ and $[\psi_n(\xi)]$ be complete sets of smooth functions that satisfy the temperature and displacement boundary conditions, respectively. Then we expand T and u in the infinite series,

$$T(\xi) = \sum_{n=1}^{\infty} T_n \phi_n(\xi), \qquad u(\xi) = \sum_{n=1}^{\infty} u_n \psi_n(\xi) \tag{7}$$

Here T_n and u_n are complex numbers. We now substitute Eq. (7) into Eqs. (5) and (6). Then we multiply Eq. (5) by $\phi_m(\xi)$ and Eq. (6) by $\psi_m(\xi)$, and we integrate both equations over the interval [0, 1]. This finally results in the infinite coupled system of linear equations,

$$\begin{bmatrix} A_{11} & B_{11} & A_{12} & B_{12} & A_{13} & B_{13} & \cdots \\ C_{11} & D_{11} & C_{12} & D_{12} & C_{13} & D_{13} & \cdots \\ A_{21} & B_{21} & A_{22} & B_{22} & A_{23} & B_{23} & \cdots \\ C_{21} & D_{21} & C_{22} & D_{22} & C_{23} & D_{23} & \cdots \\ \vdots & & & & & & \end{bmatrix} \begin{Bmatrix} T_1 \\ u_1 \\ T_2 \\ u_2 \\ \vdots \end{Bmatrix} = \begin{Bmatrix} q_1 \\ 0 \\ q_2 \\ 0 \\ \vdots \end{Bmatrix} \tag{8}$$

Here

$$A_{mn} = \int_0^1 \phi_m \phi_n'' \, d\xi - i\bar{\omega} \int_0^1 \phi_m \phi_n \, d\xi$$
$$B_{mn} = -i\bar{\omega}\bar{\delta} \int_0^1 \phi_m \psi_n' \, d\xi \tag{9}$$

$$C_{mn} = -\bar{\alpha} \int_0^1 \psi_m \phi_n' \, d\xi$$
$$D_{mn} = \int_0^1 \psi_m \psi_n'' \, d\xi + (\bar{\omega}^2 \bar{\rho} - i\bar{\omega}\bar{c}) \int_0^1 \psi_m \psi_n \, d\xi \tag{10}$$

On the right side of Eq. (8), $q_m = -\int_0^1 \phi_m q \, d\xi$ is the Fourier coefficient of q.

In practice, the infinite system of Eq. (8) is truncated and replaced by a finite, N-dimensional system of equations. In general, it may be hard to prove convergence to the exact solution as $N \to \infty$. However, if certain orthogonality conditions are satisfied,[10] then the matrix in Eq. (8) becomes block diagonal, with 2×2 blocks around the diagonal. In this case, Eq. (8) may be solved explicitly for each mode separately, and convergence can be established.[10] In the next section we consider one special case in this category.

Numerical and Analytic Investigation

We consider here two examples. In both examples we set $\bar{\delta} = 0.1$, $\bar{\alpha} = 2.15 \times 10^{-3}$, and $\bar{\rho} = 4.2 \times 10^{-16}/L^2$, which are typical to aluminum, where L is measured in meters. We also choose $L = 1$ m. We leave the damping coefficient $\bar{c}$ as a free parameter.

In the first example, the edges of the rod are fixed and are held with zero relative temperature. Thus, the boundary conditions of the problem are $T(0) = T(1) = 0$ and $u(0) = u(1) = 0$. The parabolic thermal loading $q(\xi) = q_0\xi(1-\xi)$ is applied. For the expansions in Eq. (7) we choose the basis functions $\phi_n(\xi) = \psi_n(\xi) = \sin(n\pi\xi)$, which satisfy the boundary conditions. We approximate the infinite system of Eq. (8) by a finite system with $N = 100$ equations and unknowns.

In the second example, the edges of the rod are also held with zero relative temperature but are free of any mechanical loads and kinematical constraints. Thus, the boundary conditions are $T(0) = T(1) = 0$ and $u'(0) = u'(1) = 0$. In this case we choose $\phi_n(\xi) = \sin(n\pi\xi)$ and $\psi_n(\xi) = \cos(n\pi\xi)$. Regardless of the specific form of the thermal loading q, in this case the different modes

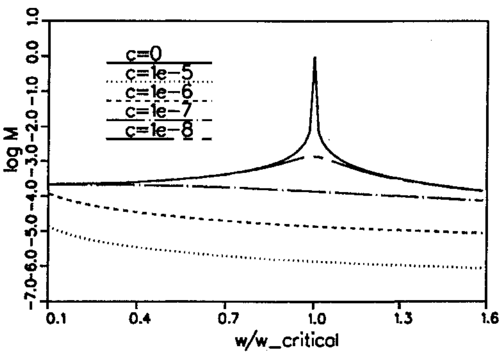

Fig. 1 The logarithm of $\mathcal{M}_k$, which measures the significance of the thermoelastic coupling effect to the temperature and displacement fields, as a function of the normalized frequency $\bar{\omega}/\bar{\omega}_{cr}$, for various values of the damping coefficient $\bar{c}$.

become uncoupled, as discussed in the previous section. Then Eq. (8) may be solved to yield the closed-form solution for the nth mode,

$$T_n = 2\frac{D_n q_n}{A_n D_n - B_n C_n}, \qquad u_n = -2\frac{C_n q_n}{A_n D_n - B_n C_n} \tag{11}$$

$$A_n = -(n\pi)^2 - i\bar{\omega}, \qquad B_n = in\pi\bar{\omega}\bar{\delta} \tag{12}$$

$$C_n = -n\pi\bar{\alpha}, \qquad D_n = -(n\pi)^2 + \bar{\omega}^2\bar{\rho} - i\bar{\omega}\bar{c} \tag{13}$$

To measure the importance of the coupling effect in each example, we will compare the absolute value of the Fourier coefficient T_n (and u_n) obtained when thermoelastic coupling effects are taken into account with that which is obtained when such effects are neglected, namely, when we set $\bar{\delta} = 0$. We denote the Fourier coefficient in the latter case by T_n^0 (and u_n^0). Thus, the relative difference between the two coefficients is measured by the two ratios,

$$\mathcal{M}_n^T \equiv \frac{|T_n - T_n^0|}{|T_n^0|}, \qquad \mathcal{M}_n^u \equiv \frac{|u_n - u_n^0|}{|u_n^0|} \tag{14}$$

In the first example, the coefficients T_n and u_n are calculated numerically by solving Eq. (8), and $\mathcal{M}_n^T$ and $\mathcal{M}_n^u$ are then evaluated from Eq. (14). However, in the second example, $\mathcal{M}_n^T$ and $\mathcal{M}_n^u$ can be obtained directly from the closed-form solution, Eqs. (11–13). A simple calculation yields in this case

$$\mathcal{M}_n \equiv \mathcal{M}_n^T = \mathcal{M}_n^u = \left| \frac{B_n C_n}{A_n D_n - B_n C_n} \right| \tag{15}$$

By carrying out the numerical calculations required in the first example and comparing the results with those of the second example, one sees that although the numerical results are different, the two examples lead to the same qualitative conclusions on the relative importance of the coupling effects. Therefore, it will be sufficient here to present the results of the second example, which is much simpler for analysis.

A case worth attention occurs when there is no damping ($\bar{c} = 0$) and $\bar{\omega}$ assumes the critical value $\bar{\omega}_{cr} = k\pi/\sqrt{\bar{\rho}}$. This value is exactly the kth eigenfrequency of the rod when no thermal effects are present and without damping. In this case we get $\mathcal{M}_k = 1$, and so the thermoelastic coupling effects are very significant.

To see this more clearly, we compare the solutions T_n and u_n in two cases. In the first case, $\bar{\omega} \to \bar{\omega}_{cr}$, $\bar{c} \to 0$, and $\bar{\delta} = 0$. Then we get $T_n = -q_n/A_n$ and $u_n = \infty$. This is the usual case of resonance, which occurs when no damping and no thermoelastic coupling effects exist and when the loading frequency is equal to the critical one. In the second case, $\bar{\omega} = \bar{\omega}_{cr}$, $\bar{c} = 0$, and $\bar{\delta} \neq 0$. Then we obtain $T_n = 0$ and $u_n = -q_n/B_n$. In this case the displacement remains finite and the temperature along the rod is identically zero! This surprising result may also be verified by considering Eqs. (5) and (6) directly.

This demonstrates the importance of the thermoelastic coupling effects in the case $\bar{\omega} = \bar{\omega}_{cr}$ and $\bar{c} = 0$; the solution changes its nature dramatically when such effects are introduced in this case. In fact, *the thermoelastic coupling precludes the appearance of resonance.*

In Fig. 1 we show $\log \mathcal{M}_k$ vs the normalized frequency $\bar{\omega}/\bar{\omega}_{cr}$ for various values of the damping coefficient $\bar{c}$. When $\bar{c} = 0$ (no damping), we obtain $\mathcal{M}_k = 1$ (or $\log \mathcal{M}_k = 0$) at $\bar{\omega} = \bar{\omega}_{cr}$, as discussed earlier. Away from this value the thermoelastic coupling effects become insignificant.

Discussion

It has been shown that a case in which thermoelastic coupling effects become very significant is that where the frequency of the thermal loading is close to the critical frequency of the rod, and externally induced viscous damping is very small. In this case, the coupling effects change the entire nature of the thermal and elastic response. Moreover, thermoelastic coupling precludes the appearance of pure elastic resonance, even with zero damping. This conclusion has been drawn in the thermally driven case; we remark that a similar conclusion may be obtained for the mechanically driven problem, where $\bar{Q} = 0$ and $\bar{F} \neq 0$ in Eqs. (3) and (4).

The case where externally induced viscous damping is virtually zero is typical to space structures, due to the lack of exterior medium in space. It can be shown that significant Fourier components with frequency ω_{cr} may be encountered for extremely long structural members (e.g., members in a large space station) in the presence of nonsmooth thermal loading (e.g., in the case of sudden exposure to solar radiation). Also, there are other types of structures that are associated with much lower critical frequencies. Although the analysis in this paper was restricted to the axial deformation of rods, it is reasonable to expect that the main conclusion may be carried over to the general case. This may be checked numerically by applying methods similar to the one used here to investigate more complicated structural models.

Thus, the inclusion of dynamic thermoelastic coupling effects in the analysis of space structures should be considered whenever the applied thermal load contains a significant Fourier component with frequency that is close to the critical frequency of the structure.

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